Electromagnetic Waves in Waveguides with Wall Impedance*

KANEYUKI KUROKAWA†, MEMBER, IRE

Summary—**A variational expression for the propagation constant of the waves in waveguides with inhomogeneous media and with wall impedance is presented. Using this expression, the shift of the propagation constant due to the wall impedance is calculated. It is also clarified how the removal of degeneracy takes place. Then the same problem is discussed using another approach, a perturbation method. The result is identical with that of the variational principle, as is to be expected. In the final section, taking degenerate TEM modes as an example, it is shown that appropriate choices of field configuration are necessary when the formula for an attenuation constant derived from the conservation of energy is applied to degenerate modes.**

I. Introduction

THE ATTENUATION constant α of waveguides due to the wall loss is usually calculated using the law of conservation of energy, which gives the formula

$$\alpha = \frac{1}{2} \frac{\text{Wall loss per unit length of waveguide}}{\text{Transmission power}}. \quad (1)$$

However, as Slater pointed out,[1] since the wall loss per unit length is not necessarily additive, when two or more waves exist simultaneously, the validity of the above formula has to be re-examined. To avoid this difficulty, an attempt has been made by various authors to solve Maxwell's equations under appropriate linear boundary conditions. Among them, Papadopoulos[2] successfully employed a perturbation method and obtained the same expression as given above for nondegenerate modes. Furthermore, he clarified how the removal of degeneracy takes place when the wall loss is finite. Independently of this work, Kurokawa[3] found a variational principle for the propagation constant of the waves in waveguides with wall impedance. The result has been published in Japanese and hence not widely circulated. Later, Collin[4] published a variational principle for the propagation constant of a TEM mode along a transmission line. However, all of these treatments are valid only when the medium is homogeneous.

The objective of this paper is to investigate the propagation of electromagnetic waves in waveguides with inhomogeneous media and with wall impedances. This is done using two different approaches; *i.e.*, a variational principle and a perturbation theorem. In Section II, the vector wave equation for waveguides with inhomogeneous media is obtained. Then, in Section III, two orthogonality relations are proved between the solutions of this equation under the boundary conditions for perfect conductor walls. These eigenfunctions can be used for the expansion of an arbitrary function and hence for the expansion of the field in waveguides with wall impedances which do not satisfy the boundary conditions for perfect conductor walls. The expansion coefficient is given at the end of Section III. Section IV gives the boundary conditions for the walls with finite impedances and Section V the variational expression for the propagation constant of the waves in waveguides with wall impedances, from which the shift of the propagation constant due to the wall impedances is calculated. Using the expansion formula obtained in Section III and the variational expression, the mechanism of the removal of degeneracy is also clarified in Section V. In Section VI, the above results are compared with those of the perturbation theorem which is slightly modified from the original expression given by Papadopoulos. Finally, in Section VII, taking degenerate TEM modes as an example, it is shown that when (1) is applied to degenerate modes, appropriate choices of field configuration are necessary. Collin neglected the possibility of degeneracy in his discussion and hence failed to clarify this point.

II. Vector Wave Equation

The values of the dielectric constant ϵ and magnetic permeability μ inside the waveguide under consideration may depend on the transverse position (x, y), but they are assumed to be independent of the longitudinal position (z). For convenience, let us resolve the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ into two components, the transverse component (subscript t) and longitudinal component (subscript z). Then we have

$$\boldsymbol{E} = (\boldsymbol{E}_t + \boldsymbol{k}E_z)e^{j\omega t - \gamma z}$$
$$\boldsymbol{H} = (\boldsymbol{H}_t + \boldsymbol{k}H_z)e^{j\omega t - \gamma z} \quad (2)$$

where $\boldsymbol{k}$ is a unit vector in the longitudinal direction and γ is the propagation constant. Substituting (2) into the equations

$$\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H}, \qquad \nabla \times \boldsymbol{H} = j\omega\epsilon\boldsymbol{E} \quad (3)$$

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† Institute of Industrial Science, University of Tokyo, Japan.

[1] J. C. Slater, "Microwave Electronics," D. Van Nostrand Co., Inc., New York, N. Y., p. 49; 1950.

[2] V. M. Papadopoulos, "Propagation of electromagnetic waves in cylindrical waveguides with imperfectly conducting walls," *Quart. J. Mech. Appl. Math.*, vol. 7, pp. 325–334; September, 1954.

[3] K. Kurokawa, "The effect of wall losses on the propagation constant of waveguides," *J. Elec. Commun. Engrs. Japan*, vol. 39, pp. 794–800; September, 1956.

[4] R. E. Collin, "A variational integral for propagation constant of lossy transmission lines," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 339–342; May, 1960.

we obtain four equations:

$$\begin{aligned} \nabla \times \boldsymbol{E}_t &= -j\omega\mu k H_z \\ \nabla \times \boldsymbol{H}_t &= j\omega\epsilon k E_z \\ \gamma \boldsymbol{k} \times \boldsymbol{E}_t + \boldsymbol{k} \times \nabla E_z &= j\omega\mu \boldsymbol{H}_t \\ \gamma \boldsymbol{k} \times \boldsymbol{H}_t + \boldsymbol{k} \times \nabla H_z &= -j\omega\epsilon \boldsymbol{E}_t. \end{aligned} \tag{4}$$

Next, substituting (2) into the equations

$$\nabla \cdot \epsilon \boldsymbol{E} = 0, \qquad \nabla \cdot \mu \boldsymbol{H} = 0 \tag{5}$$

which are obtained from (3), we have

$$\nabla \cdot \epsilon \boldsymbol{E}_t = \gamma \epsilon E_z, \qquad \nabla \cdot \mu \boldsymbol{H}_t = \gamma \mu H_z. \tag{6}$$

From (4) and (6), we eliminate E_z, H_z, $\boldsymbol{H}_t$. The result is a vector wave equation for $\boldsymbol{E}_t$:

$$\mu \nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_t - \nabla \frac{1}{\epsilon} \nabla \cdot \epsilon \boldsymbol{E}_t - (\omega^2 \epsilon \mu + \gamma^2) \boldsymbol{E}_t = 0 \qquad (\text{in } S), \tag{7}$$

where S means the cross section of the waveguide. The boundary condition for perfect conductor walls is given by the following two equations:

$$\nabla \cdot \epsilon \boldsymbol{E}_t = 0, \qquad \boldsymbol{n} \times \boldsymbol{E}_t = 0 \qquad (\text{on } l), \tag{8}$$

where l means the periphery of S and $\boldsymbol{n}$ is an outer normal unit vector. For the ideal waveguide, solving (7) under the condition (8) and inserting the result into (4) and (6), all the field components can be obtained. If we eliminate E_z, H_z and $\boldsymbol{E}_t$ instead of E_z, H_z and $\boldsymbol{H}_t$ from (4) and (6), we have a vector wave equation for $\boldsymbol{H}_t$ similar to (7). However, the discussion using $\boldsymbol{H}_t$ goes in parallel with the one using $\boldsymbol{E}_t$, giving no new results, hence we shall concentrate on $\boldsymbol{E}_t$ only.

III. Orthogonality Relations

There are two orthogonality relations between the solutions of (7) which satisfy the condition (8). To derive these relations, let us write one of the solutions as $\boldsymbol{E}_{tm}$ and another as $\boldsymbol{E}_{tn}$. The corresponding propagation constants will be written as γ_m and γ_n, respectively. Since $\boldsymbol{E}_{tm}$ satisfies (7),

$$\mu \nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tm} - \nabla \frac{1}{\epsilon} \nabla \cdot \epsilon \boldsymbol{E}_{tm} - (\omega^2 \epsilon \mu + \gamma_m^2) \boldsymbol{E}_{tm} = 0 \qquad (\text{in } S).$$

Taking a scalar product with

$$\left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tn} - \omega^2 \epsilon \boldsymbol{E}_{tn} \right)$$

and integrating over the cross section S, we obtain

$$\begin{aligned} &\int \mu \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tm} - \omega^2 \epsilon \boldsymbol{E}_{tm} \right) \cdot \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tn} - \omega^2 \epsilon \boldsymbol{E}_{tn} \right) dS \\ &- \int \omega^2 \frac{1}{\epsilon} (\nabla \cdot \epsilon \boldsymbol{E}_{tm})(\nabla \cdot \epsilon \boldsymbol{E}_{tn}) dS \\ &- \gamma_m^2 \left\{ \int \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tm} \cdot \nabla \times \boldsymbol{E}_{tn} dS - \int \omega^2 \epsilon \boldsymbol{E}_{tm} \cdot \boldsymbol{E}_{tn} dS \right\} = 0 \end{aligned} \tag{9}$$

where we used (8). Interchanging the subscripts m and n and subtracting the result from (9), we have

$$(\gamma_m^2 - \gamma_n^2) \left\{ \int \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tm} \cdot \nabla \times \boldsymbol{E}_{tn} dS - \int \omega^2 \epsilon \boldsymbol{E}_{tm} \cdot \boldsymbol{E}_{tn} dS \right\} = 0.$$

If $\gamma_m^2 \neq \gamma_n^2$, then

$$\int \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tm} \cdot \nabla \times \boldsymbol{E}_{tn} dS - \int \omega^2 \epsilon \boldsymbol{E}_{tm} \cdot \boldsymbol{E}_{tn} dS = 0. \tag{10}$$

This is equivalent to the orthogonality relation[5]

$$\int \boldsymbol{k} \cdot \boldsymbol{E}_{tm} \times \boldsymbol{H}_{tn} dS = 0 \tag{11}$$

which has been derived by Adler[6] using a different approach. To check the relation (11), using (4) we write $\boldsymbol{H}_{tn}$ in terms of $\boldsymbol{E}_{tn}$:

$$\boldsymbol{k} \times \boldsymbol{H}_{tn} = \frac{-1}{j\omega\gamma_n} \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tn} - \omega^2 \epsilon \boldsymbol{E}_{tn} \right). \tag{12}$$

Combination of (11) with (12) gives

$$\begin{aligned} &\int \boldsymbol{k} \cdot \boldsymbol{E}_{tm} \times \boldsymbol{H}_{tn} dS \\ &= \frac{1}{j\omega\gamma_n} \left\{ \int \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tm} \cdot \nabla \times \boldsymbol{E}_{tn} dS - \int \omega^2 \epsilon \boldsymbol{E}_{tm} \cdot \boldsymbol{E}_{tn} dS \right\} = 0 \end{aligned}$$

which is equivalent to (10).

In case (10) is satisfied, from (9) we have

$$\begin{aligned} &\int \mu \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tm} - \omega^2 \epsilon \boldsymbol{E}_{tm} \right) \cdot \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tn} - \omega^2 \epsilon \boldsymbol{E}_{tn} \right) dS \\ &- \int \omega^2 \frac{1}{\epsilon} \nabla \cdot \epsilon \boldsymbol{E}_{tm} \nabla \cdot \epsilon \boldsymbol{E}_{tn} dS = 0. \end{aligned} \tag{13}$$

[5] The integrand is not $\boldsymbol{k} \cdot \boldsymbol{E}_{tm} \times \boldsymbol{H}_{tn}^*$, hence (11) is not necessarily equivalent to the orthogonality relation between powers carried by each mode.

[6] R. B. Adler, "Waves on inhomogeneous cylincrical structures," Proc. IRE, vol. 40, pp. 339–348; March, 1952.

This is another orthogonality relation between $\boldsymbol{E}_{tm}$ and $\boldsymbol{E}_{tn}$. As we can easily see from (4), (6) and (12), (13) corresponds to the orthogonality relation between the $\boldsymbol{H}_{tn}$'s

$$\int \frac{1}{\epsilon} \nabla \times \boldsymbol{H}_{tm} \cdot \nabla \times \boldsymbol{H}_{tn} dS - \int \omega^2 \mu \boldsymbol{H}_{tm} \cdot \boldsymbol{H}_{tn} dS = 0 \quad (14)$$

which is a dual form of (10).

In case $\boldsymbol{E}_{tm}$ and $\boldsymbol{E}_{tn}$ are degenerate, $\gamma_m^2 = \gamma_n^2$ and the orthogonality relation (10) does not necessarily hold. However, it is always possible to choose the modes so as to satisfy (10). Further, the derivation of (14) guarantees that, if the $\boldsymbol{E}_{tn}$'s are selected to be orthogonal to each other, the corresponding $\boldsymbol{H}_{tn}$'s become automatically orthogonal to each other and vice versa.

The above orthogonality relations will be used several times later and, since their expressions are lengthy, we shall introduce abbreviations $P(m, n)$ and $Q(m, n)$ given by

$$P(m, n) = \int \mu \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tm} - \omega^2 \epsilon \boldsymbol{E}_{tm} \right) \cdot \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tn} - \omega^2 \epsilon \boldsymbol{E}_{tn} \right) dS - \int \omega^2 \frac{1}{\epsilon} \nabla \cdot \epsilon \boldsymbol{E}_{tm} \nabla \cdot \epsilon \boldsymbol{E}_{tn} dS \quad (15)$$

$$Q(m, n) = \int \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tm} \cdot \nabla \times \boldsymbol{E}_{tn} dS - \int \omega^2 \epsilon \boldsymbol{E}_{tm} \cdot \boldsymbol{E}_{tn} dS. \quad (16)$$

When expanding an arbitrary function $\boldsymbol{E}_{t\alpha}$ in terms of the $\boldsymbol{E}_{tn}$'s as[7]

$$E_{t\alpha} = \sum_n c_n \boldsymbol{E}_{tn} \quad (17)$$

we multiply the both sides of (17) by

$$\left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tn} - \omega^2 \epsilon \boldsymbol{E}_{tn} \right),$$

and integrate over S. Using the equation

$$\int \boldsymbol{E}_{t\alpha} \cdot \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tn} - \omega^2 \epsilon \boldsymbol{E}_{tn} \right) dS = Q(\alpha, n) - \int \boldsymbol{n} \times \boldsymbol{E}_{t\alpha} \cdot \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tn} dl \quad (18)$$

and the orthogonality relation

$$Q(m, n) = 0 \qquad (m \neq n)$$

we can determine the expansion coefficients c_n's:

$$c_n = \frac{1}{Q(n, n)} \left\{ Q(\alpha, n) - \int \boldsymbol{n} \times \boldsymbol{E}_{t\alpha} \cdot \frac{1}{\mu} \nabla \times \boldsymbol{E}_{tn} dl \right\}. \quad (19)$$

One may be tempted to multiply both sides of (17) by $\epsilon \boldsymbol{E}_{tn}$ and integrate over S. However, this gives no useful relations to determine c_n's, since in general

$$\int \epsilon \boldsymbol{E}_{tm} \cdot \boldsymbol{E}_{tn} dS \neq 0.$$

IV. Wall Impedances

In the previous sections, the waveguide walls are assumed to be perfect conductors. Let us now introduce wall impedances Z_1 and Z_2 defined by

$$\left. \begin{aligned} Z_1 \boldsymbol{H}_t &= \boldsymbol{n} \times k E_z \\ Z_2 k H_z &= \boldsymbol{n} \times \boldsymbol{E}_t \end{aligned} \right\} \text{(on } l\text{)}. \quad (20)$$

Z_1 and Z_2 may be two different functions of the position along l but they are assumed to be independent of z. Z_1 represents the wall impedance against the current flowing in the longitudinal direction and Z_2 in the transverse direction. Using (4) and (6), (20) can be rewritten in terms of $\boldsymbol{E}_t$ only:

$$\boldsymbol{n} \frac{1}{\epsilon} \nabla \cdot \epsilon \boldsymbol{E}_t = - \frac{Z_1}{j\omega} \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_t - \omega^2 \epsilon \boldsymbol{E}_t \right) \qquad \text{(on } l\text{)} \quad (21)$$

$$\boldsymbol{n} \times \boldsymbol{E}_t = - \frac{Z_2}{j\omega} \frac{1}{\mu} \nabla \times \boldsymbol{E}_t.$$

Therefore, our problem is reduced to that of solving (7) under the boundary condition (21).

If we set both Z_1 and Z_2 equal to zero, (21) becomes (8) which corresponds to perfect conductor walls. In case the waveguide has ordinary conductor walls, Z_1 and Z_2 become the characteristic impedance of the wall material.

V. Variational Principle for γ^2

Let us consider the relevance

$$\gamma^2 = \frac{\int \mu \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_t - \omega^2 \epsilon \boldsymbol{E}_t \right)^2 dS - \int \omega^2 \frac{1}{\epsilon} (\nabla \cdot \epsilon \boldsymbol{E}_t)^2 dS + \int \frac{Z_1}{j\omega} \left(\nabla \times \frac{1}{\mu} \nabla \times \boldsymbol{E}_t - \omega^2 \epsilon \boldsymbol{E}_t \right)^2 dl}{\int \frac{1}{\mu} (\nabla \times \boldsymbol{E}_t)^2 dS - \int \omega^2 \epsilon \boldsymbol{E}_t^2 dS + \int \frac{Z_2}{j\omega} \left(\frac{1}{\mu} \nabla \times \boldsymbol{E}_t \right)^2 dl} \quad (22)$$

and take the variation. A little manipulation shows that

[7] $\boldsymbol{n} \times \boldsymbol{E}_{t\alpha}$ is not necessarily zero and $\boldsymbol{E}_{t\alpha}$ can be a field function in a waveguide with wall impedances.

$$\delta\gamma^2\left\{\int \frac{1}{\mu}(\nabla\times E_t)^2dS-\int \omega^2\epsilon E_t^2dS\right\}$$
$$=2\int\left\{\mu\nabla\times\frac{1}{\mu}\nabla\times E_t-\nabla\frac{1}{\epsilon}\nabla\cdot\epsilon E_t-(\omega^2\epsilon\mu+\gamma^2)E_t\right\}$$
$$\cdot\left(\nabla\times\frac{1}{\mu}\nabla\times\delta E_t-\omega^2\epsilon\delta E_t\right)dS$$
$$+2\int\left\{n\frac{1}{\epsilon}\nabla\cdot\epsilon E_t+\frac{Z_1}{j\omega}\left(\nabla\times\frac{1}{\mu}\nabla\times E_t-\omega^2\epsilon E_t\right)\right\}$$
$$\cdot\left(\nabla\times\frac{1}{\mu}\nabla\times\delta E_t-\omega^2\epsilon\delta E_t\right)dl$$
$$-2\gamma^2\int\left(n\times E_t+\frac{Z_2}{j\omega}\frac{1}{\mu}\nabla\times E_t\right)\cdot\frac{1}{\mu}\nabla\times\delta E_t dl. \quad (23)$$

If E_t satisfies both (7) and (21), then the right-hand side of (23) becomes zero and hence $\delta\gamma^2$ becomes zero provided that

$$\int \frac{1}{\mu}(\nabla\times E_t)^2dS-\int \omega^2\epsilon E_t^2dS \neq 0. \quad (24)$$

Furthermore, if the first-order variation $\delta\gamma^2$ of the expression (22) is zero for all possible deviation δE_t from a certain E_t, then from (23) we see that such an E_t satisfies (7) and (21) simultaneously. Thus we conclude that (22) is an appropriate variational expression for γ^2.

Now let us assume that the magnitudes of Z_1 and Z_2 are both small and that there exists a solution E_{ta} of (7) and (21) in the vicinity of E_{tp}, one of the eigenfunctions E_{tn}'s of the ideal waveguide. Then, substituting E_{tp} in (22), we obtain the first-order approximation of the propagation constant γ_a of E_{ta} through the relation

$$\gamma_\alpha^2 \approx \frac{P(p,p)+Z_1(p,p)}{Q(p,p)+Z_2(p,p)}$$
$$\approx \frac{P(p,p)}{Q(p,p)}+\frac{Z_1(p,p)}{Q(p,p)}-\frac{P(p,p)Z_2(p,p)}{Q(p,p)Q(p,p)}$$
$$=\gamma_p^2+\frac{1}{Q(p,p)}\left\{Z_1(p,p)-\gamma_p^2Z_2(p,p)\right\} \quad (25)$$

where $Z_1(p, p)$ and $Z_2(p, p)$ are given by setting both m and n equal to p in the expressions

$$Z_1(m,n)=\int \frac{Z_1}{j\omega}\left(\nabla\times\frac{1}{\mu}\nabla\times E_{tm}-\omega^2\epsilon E_{tm}\right)$$
$$\cdot\left(\nabla\times\frac{1}{\mu}\nabla\times E_{tn}-\omega^2\epsilon E_{tn}\right)dl \quad (26)$$

and

$$Z_2(m,n)=\int \frac{Z_2}{j\omega}\left(\frac{1}{\mu}\nabla\times E_{tm}\right)\cdot\left(\frac{1}{\mu}\nabla\times E_{tn}\right)dl \quad (27)$$

respectively. The second term on the right-hand side of (25) corresponds to the shift of the propagation constant due to the wall impedances. The solution E_{ta} can be expressed as a linear combination of the eigenfunctions E_{tn}'s of the ideal waveguide provided that they are complete:

$$E_{ta}=E_{tp}+\sum_{n\neq p} c_nE_{tn}. \quad (28)$$

To determine c_n, we substitute $E_{ta}+\delta c_nE_{tn}$ in (22).[8] Since E_{ta} is a solution of (7) and (21), the variation $\delta\gamma^2$ due to δc_nE_{tn} must be zero. Thus, we have

$$\delta\gamma^2=2\left\{\frac{P(\alpha,n)}{Q(\alpha,\alpha)}+\frac{Z_1(\alpha,n)}{Q(\alpha,\alpha)}-\frac{P(\alpha,\alpha)Q(\alpha,n)}{Q(\alpha,\alpha)Q(\alpha,\alpha)}-\frac{P(\alpha,\alpha)Z_2(\alpha,n)}{Q(\alpha,\alpha)Q(\alpha,\alpha)}\right\}\delta c_n=0. \quad (29)$$

Using the relations

$$P(\alpha,n)=\gamma_n^2Q(\alpha,n)$$
$$P(\alpha,\alpha)\approx\gamma_p^2Q(\alpha,\alpha)$$

which can be obtained from (9), (29) shows that

$$Q(\alpha,n)\approx\frac{1}{\gamma_p^2-\gamma_n^2}\left\{Z_1(p,n)-\gamma_p^2Z_2(p,n)\right\}. \quad (30)$$

Combination of (19) with (30) gives

$$c_n=\frac{Q(\alpha,n)+Z_2(\alpha,n)}{Q(n,n)}$$
$$\approx\frac{1}{Q(n,n)}\frac{1}{\gamma_p^2-\gamma_n^2}\left\{Z_1(p,n)-\gamma_n^2Z_2(p,n)\right\}. \quad (31)$$

If $\gamma_p^2=\gamma_n^2$, that is, if the modes n and p are degenerate the corresponding c_n given by (31) becomes infinite.[9] Therefore, we have to employ a different approach. For simplicity, first let us consider the case of two-fold degeneracy. Let E_{ta} and E_{tb} be the degenerate modes which are orthogonal to each other and of which the magnitudes are so chosen the $Q(a, a)=Q(b, b)$. Now we assume an appropriate form of the solutions of (7) and (21) as

$$E_t=AE_{ta}+BE_{tb}$$
$$+\text{(first- and higher-order terms of } Z_1 \text{ and } Z_2) \quad (32)$$

and determine the coefficient A and B. Since only the ratio between A and B is to be determined, we impose the condition $A^2+B^2=1$. Substituting (32) in (22) and using Lagrange's method of multipliers to obtain the condition $\delta\gamma^2=0$, we have

$$A=\cos\theta, \qquad B=\sin\theta \quad (33)$$

[8] Do ot substitute the right-hand side of (28) in (22) directly, since the term-by-term differentation does not necessarily give the correct answer.

[9] When the frequency approaches the critical value beyond which the cutoff phenomena takes place, $Q(n, n)$ approaches zero. However, a little manipulation shows that the limiting value of the right-hand side of (31) remains finite.

and

$$A = -\sin\theta, \qquad B = \cos\theta, \tag{34}$$

where θ is given through the relation

$$\tan 2\theta = \frac{2\{Z_1(a, b) - \gamma_p^2 Z_2(a, b)\}}{\{Z_1(a, a) - Z_1(b, b)\} - \gamma_p^2\{Z_2(a, a) - Z_2(b, b)\}}. \tag{35}$$

Thus we see that the correct solutions $\boldsymbol{E}_{t\alpha}$ and $\boldsymbol{E}_{t\beta}$ must be in the vicinity of

$$\boldsymbol{E}_{tp} = \cos\theta \boldsymbol{E}_{ta} + \sin\theta \boldsymbol{E}_{tb} \tag{36}$$

and

$$\boldsymbol{E}_{tq} = -\sin\theta \boldsymbol{E}_{ta} + \cos\theta \boldsymbol{E}_{tb} \tag{37}$$

respectively. The propagation constant of each mode is given by (25), provided that $\boldsymbol{E}_{tp}$ in (25) is replaced by (36) and (37) respectively. In some cases, the right-hand side of (35) becomes undetermined. This means that the first-order removal of degeneracy due to the wall impedances does not take place and that an arbitrary value can be assigned for θ. TE_{11} modes in circular waveguides with uniform wall impedances is an example.

Next, assuming the completeness of the $\boldsymbol{E}_{tn}$'s, we expand $\boldsymbol{E}_{t\alpha}$ and $\boldsymbol{E}_{t\beta}$ as follows:

$$E_{t\alpha} = \boldsymbol{E}_{tp} + \sum_{n\neq a,b} c_n' \boldsymbol{E}_{tn} \tag{38}$$

$$\boldsymbol{E}_{t\beta} = \boldsymbol{E}_{tq} + \sum_{n\neq a,b} c_n'' \boldsymbol{E}_{tn}. \tag{39}$$

Then the expansion coefficients c_n' and c_n'' must be small in magnitude. c_n' and c_n'' are given by (31) provided that the $\boldsymbol{E}_{tp}$ is replaced by (36) and (37) respectively. $\boldsymbol{E}_{tp}$ and $\boldsymbol{E}_{tq}$ defined by (36) and (37) are orthogonal to each other, *i.e.*,

$$Q(p, q) = P(p, q) = 0. \tag{40}$$

Furthermore, they satisfy the following equation:

$$Z_1(p, q) - \gamma_p^2 Z_2(p, q) = 0. \tag{41}$$

As we see from (31), (41) means that there is no coupling between $\boldsymbol{E}_{tp}$ and $\boldsymbol{E}_{tq}$ through the wall impedances. Any other combination of $\boldsymbol{E}_{ta}$ and $\boldsymbol{E}_{tb}$ cannot satisfy this condition. Threfore, we cannot choose such a field as an approximate form of a mode which has an independent propagation constant.

In the case of multiple degeneracy, we assume the form of solutions as follows:

$$\boldsymbol{E}_t = A\boldsymbol{E}_{ta} + B\boldsymbol{E}_{tb} + C\boldsymbol{E}_{tc} + \cdots$$
$$+ \text{(first- and higher-order terms of } Z_1 \text{ and } Z_2).$$

Under the condition $A^2+B^2+C^2+\cdots=1$, the coefficients $A, B, C, \cdots$ will be determined so as to satisfy $\delta\gamma^2=0$. The rest of the argument given above for two-fold degeneracy does hold equally well in this case and no further discussion may be required.

VI. Perturbation Theorem

In this section we shall consider a perturbation method and compare the result with that of Section V. Assuming the existence of a solution $\boldsymbol{E}_{t\alpha}$ of (7) and (21) in the vicinity of $\boldsymbol{E}_{tp}$, one of the eigenfunctions $\boldsymbol{E}_{tn}$'s for the ideal waveguide, we shall first calculate the propagation constant. Since $\boldsymbol{E}_{t\alpha}$ is a solution of (7),

$$\mu\nabla \times \frac{1}{\mu}\nabla \times \boldsymbol{E}_{t\alpha} - \nabla\frac{1}{\epsilon}\nabla\cdot\epsilon\boldsymbol{E}_{t\alpha} - (\omega^2\epsilon\mu + \gamma_\alpha^2)\boldsymbol{E}_{t\alpha} = 0.$$

Multiplying by

$$\left(\nabla \times \frac{1}{\mu}\nabla \times \boldsymbol{E}_{tn} - \omega^2\epsilon\boldsymbol{E}_{tn}\right),$$

and integrating over S, we have

$$\gamma_\alpha^2 Q(\alpha, n) + P(\alpha, n) + Z_1(\alpha, n) - \gamma_\alpha^2 Z_2(\alpha, n) = 0 \tag{42}$$

where we used (21). If we interchange the subscripts α and n before the integration is performed, we have

$$\gamma_n^2 Q(\alpha, n) + P(\alpha, n) = 0. \tag{43}$$

Subtraction of (43) from (42) gives

$$(\gamma_\alpha^2 - \gamma_n^2)Q(\alpha, n) = Z_1(\alpha, n) - \gamma_\alpha^2 Z_2(\alpha, n). \tag{44}$$

Let us first set n equal to p. Since $\boldsymbol{E}_{tp} \approx \boldsymbol{E}_{t\alpha}$, we obtain the same expression as (25):

$$\gamma_\alpha^2 - \gamma_p^2 = \frac{1}{Q(\alpha, p)}\{Z_1(\alpha, p) - \gamma_\alpha^2 Z_2(\alpha, p)\}$$
$$\approx \frac{1}{Q(p, p)}\{Z_1(p, p) - \gamma_p^2 Z_2(p, p)\}. \tag{45}$$

When $n \neq p$, (44) gives

$$Q(\alpha, n) = \frac{1}{\gamma_\alpha^2 - \gamma_n^2}\{Z_1(\alpha, n) - \gamma_\alpha^2 Z_2(\alpha, n)\}$$
$$\approx \frac{1}{\gamma_p^2 - \gamma_n^2}\{Z_1(p, n) - \gamma_p^2 Z_2(p, n)\} \tag{46}$$

which is to be compared with (30).

When two modes $\boldsymbol{E}_{ta}$ and $\boldsymbol{E}_{tb}$ are degenerate, we replace $\boldsymbol{E}_{t\alpha}$ in (44) by (32). Setting n equal to a and to b, we have

$$\left.\begin{aligned}(\gamma_\alpha^2 - \gamma_p^2)AQ(a, a) &= A\{Z_1(a, a) - \gamma_p^2 Z_2(a, a)\}\\ &\quad + B\{Z_1(b, a) - \gamma_p^2 Z_2(b, a)\}\\ (\gamma_\alpha^2 - \gamma_p^2)BQ(b, b) &= A\{Z_1(b, a) - \gamma_p^2 Z_2(b, a)\}\\ &\quad + B\{Z_1(b, b) - \gamma_p^2 Z_2(b, b)\}\end{aligned}\right\}. \tag{47}$$

The condition that at least one of the coefficients A and B has nonzero value gives (33), (34) and (35). Thus we see that the results obtained by the variational principle and the perturbation theorem are identical, as is to be expected.

The perturbation method originally developed by Papadopoulos is based on the orthogonality relation

$$\int \boldsymbol{E}_{tm} \cdot \boldsymbol{E}_{tn} dS = 0 \qquad (m \neq n)$$

which does hold between the modes in waveguides with homogeneous media. However, its extension does not exist, as we explained at the end of Section III. Therefore, in the above discussion, we had to use $Q(m, n) = 0$, $(m \neq n)$ which is equivalent to (11).

VII. Example

As an example of the removal of degeneracy, let us consider the waves traveling along two parallel wires above ground. The configuration is shown in Fig. 1. If we assume that the wires as well as the ground are perfect conductors, then the fields $\boldsymbol{E}_t$ and $\boldsymbol{H}_t$ are expressed in terms of a scalar potential function ϕ which satisfies Laplace's equation $\nabla^2\phi = 0$ (in S) and the boundary condition $\phi = \text{constant}$ (on l) (for each conductor we can assign a different value):

$$\boldsymbol{E}_t = \nabla\phi, \qquad \boldsymbol{H}_t = \frac{1}{Z_0} \boldsymbol{k} \times \nabla\phi \tag{48}$$

where

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}},$$

ϵ_0 and μ_0 are the dielectric constant and magnetic permeability of the space, respectively. Since there are two wires whose potential can be specified independently, there are two independent functions ϕ and correspondingly two independent waves. Their propagation constants are the same and given by

$$\gamma_0 = j\omega\sqrt{\mu_0\epsilon_0}.$$

The deviation $\Delta\gamma$ of the actual propagation constant γ from γ_0 due to the wall impedances can be calculated by inserting $\boldsymbol{E}_t$ of (48) in (22). The final result is

$$\Delta\gamma = \frac{\gamma^2 - \gamma_0^2}{2\gamma} = \frac{1}{2Z_0} \frac{\int Z_1 \left(\frac{\partial\phi}{\partial n}\right)^2 dl}{\int (\nabla\phi)^2 dS}. \tag{49}$$

Eq. (49) can be rewritten in terms of the currents I_1 and I_2 flowing through the wires:

$$\Delta\gamma = \frac{Z_e}{2\pi} \frac{A_{11}I_1^2 + 2A_{12}I_1I_2 + A_{22}I_2^2}{Z_{11}I_1^2 + 2Z_{12}I_1I_2 + Z_{22}I_2^2}, \tag{50}$$

where

$$\left.\begin{aligned} Z_{11} &= Z_{22} \approx \frac{Z_0}{2\pi} \log \frac{2h}{r} \\ Z_{12} &= Z_{21} \approx \frac{Z_0}{2\pi} \log \frac{d}{c} \\ A_{11} &= A_{22} \approx \frac{1}{2h} + \frac{Z_c}{Z_e} \frac{1}{2r} \\ A_{12} &= A_{21} \approx \frac{2h}{d^2}. \end{aligned}\right\}. \tag{51}$$

Z_e and Z_c are the characteristic impedances of the ground and wires, respectively.

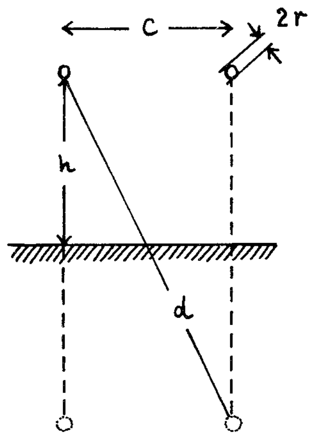

Fig. 1—Wire configuration.

Since the original waves are degenerate, for the calculation of $\Delta\gamma$ the ratio between I_1 and I_2 must be chosen so that γ^2, hence $\Delta\gamma$, has a stationary value. The method of undetermined multiplier shows that this condition is satisfied when

$$I_2 = -I_1 \tag{52}$$

and when

$$I_2 = I_1. \tag{53}$$

The corresponding $\Delta\gamma$'s are given by

$$\Delta\gamma_1 = \frac{Z_e}{2\pi} \frac{A_{11} - A_{12}}{Z_{11} - Z_{12}} \tag{54}$$

and

$$\Delta\gamma_2 = \frac{Z_e}{2\pi} \frac{A_{11} + A_{12}}{Z_{11} + Z_{12}} \tag{55}$$

respectively.

When arbitrary currents I_1 and I_2 are flowing through the wires, they must be interpreted as shown in Fig. 2, *i.e.*, the sum of the two independent modes given by (52) and (53). Since the propagation constants of these

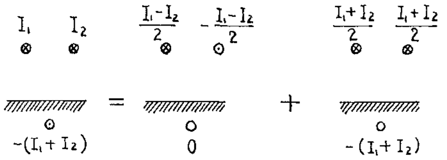

Fig. 2—Decomposition into two independent modes.

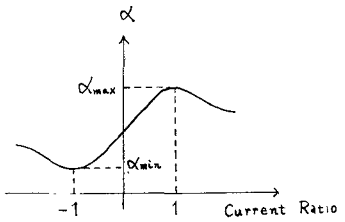

Fig. 3—α given by (1) vs current ratio.

modes are not the same, the current ratio, hence the field configuration, changes as the observation point moves along the wires. Only the currents satisfying either (52) or (53) can keep their ratio unchanged.

If the current ratio is real, the real part of the right-hand side of (50) gives exactly the same expression as that of (1). Depending on the current ratio, therefore, (1) gives a value something like Fig. 3. However, only two values, α_{max} and α_{min}, give the true attenuation constants corresponding to the independent modes discussed above. Thus, we see that a special care must be taken when (1) is applied to degenerate modes, even though they are TEM modes. Collin[4] neglected the possibility of degeneracy in his discussion and hence failed to clarify the point mentioned above.

VIII. Conclusion

A variational expression for the propagation constant of the waves in waveguides with inhomogeneous media and with wall impedance is presented. Using this expression, the shift of the propagation constant due to the wall impedances is calculated. It is also clarified how the removal of degeneracy takes place. The result is compared with that of a perturbation theorem. In Section VII, taking degenerate TEM modes as an example, it is shown that appropriate choices of field configuration are necessary when the formula for the attenuation constant derived from the conservation of energy is applied to degenerate modes.

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